# THE OPERATOR FORMULA FOR MONOTONE TRIANGLES – SIMPLIFIED PROOF AND THREE GENERALIZATIONS

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ABSTRACT. We provide a simplified proof of our operator formula for the number of monotone triangles with prescribed bottom row, which enables us to deduce three generalizations of the formula. One of the generalizations concerns a certain weighted enumeration of monotone triangles which specializes to the weighted enumeration of alternating sign matrices with respect to the number of $-1$s in the matrix when prescribing $(1, 2, \ldots, n)$ as the bottom row of the monotone triangle.

## 1. INTRODUCTION

A *monotone triangle* is a triangular array of integers of the following form

$$
\begin{array}{ccccccccccc}
& & & & & a_{n,n} & & & & & \\
& & & & a_{n-1,n-1} & & a_{n-1,n} & & & & \\
& & & \cdots & & \cdots & & \cdots & & & \\
& & a_{3,3} & & \cdots & & \cdots & & a_{3,n} & & \\
& a_{2,2} & & a_{2,3} & & \cdots & & \cdots & & a_{2,n} & \\
a_{1,1} & & a_{1,2} & & a_{1,3} & & \cdots & & \cdots & & a_{1,n}
\end{array}
,
$$

which is monotone increasing in northeast and in southeast direction and strictly increasing along rows, that is $a_{i,j} \le a_{i+1,j+1}$ for $1 \le i \le j < n$, $a_{i,j} \le a_{i-1,j}$ for $1 < i \le j \le n$ and $a_{i,j} < a_{i,j+1}$ for $1 \le i \le j \le n-1$. Monotone triangles with bottom row $(1, 2, \ldots, n)$ correspond to $n \times n$ alternating sign matrices, the enumeration of which (there are exactly $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ of them) provided an open problem for quite some time, see [1].

In [3] we have shown that the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$ is given by

$$
\prod_{1 \le s < t \le n} (\operatorname{id} - E_{k_s} + E_{k_s} E_{k_t}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i},
\tag{1.1}
$$

where $E_x$ denotes the shift operator, defined as $E_x p(x) = p(x + 1)$. In such an "operator formula", the product of operators is understood as the composition. Moreover note that the shift operators with respect to different variables commute and, consequently, it does not matter in which order the operators in the product are applied. This formula was the basis for our new proof of the refined alternating sign matrix theorem [4].

The purpose of this article is to present a (very much) simplified proof of (1.1) and three generalizations that arise quite naturally out of this new proof. Next we describe these generalizations.

Regarding the first generalization consider the following inverse question. Given a polynomial $r(X, Y)$ in $X, X^{-1}, Y, Y^{-1}$ (e.g., over $\mathbb{C}$), find a combinatorial interpretation for the numbers

$$\prod_{1 \le s < t \le n} r(E_{k_s}, E_{k_t}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}. \tag{1.2}$$

For the polynomial $r(X, Y) = \mathrm{id} - X + XY$, a combinatorial interpretation is obviously given by monotone triangles with prescribed bottom row; for the polynomial $r(X, Y) = Y$, a combinatorial interpretation is given by Gelfand–Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$ (which are almost defined as monotone triangles only they need not necessarily strictly increase along rows) as the number of these objects is given by

$$\prod_{1 \le i < j \le n} \frac{k_j - k_i + j - i}{j - i}, \tag{1.3}$$

see, for instance, [2]. Note that Gelfand–Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$ are in bijection with semistandard tableaux of shape $(k_n, k_{n-1}, \ldots, k_1)$. Here, we present a combinatorial interpretation of (1.2) for all polynomials $r(X, Y)$ of the form

$$r(X, Y) = Y + (X - \mathrm{id})(Y - \mathrm{id})s(X, Y), \tag{1.4}$$

where $s(X, Y)$ is an arbitrary polynomial in $X, X^{-1}, Y, Y^{-1}$. (Obviously, (1.1) is the special case $s(X, Y) = 1$ and (1.3) is the special case $s(X, Y) = 0$.) Notably, this interpretation is not only valid for $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \ldots < k_n$, but also for general $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and thus this will also lead to an interpretation of (1.1) for all $(k_1, \ldots, k_n) \in \mathbb{Z}^n$.

The second generalization concerns the weighted enumeration of alternating sign matrices with respect to the number of $-1$s in the matrix, which was introduced by Mills, Robbins and Rumsey, see [8]. A $-1$ in the alternating sign matrix translates into an entry $a_{i,j}$ with $i \ne 1$ and $a_{i-1,j-1} < a_{i,j} < a_{i-1,j}$ in the corresponding monotone triangle and, therefore, we define the $Q$–weight of a monotone triangle as $Q$ raised to the power of the number of such entries.

**Theorem 1.** *The generating function of monotone triangles with prescribed bottom row* $(k_1, \ldots, k_n)$, $k_1 < k_2 < \cdots < k_n$, *and with respect to the* $Q$–*weight is given by*

$$\prod_{1 \le s < t \le n} (\mathrm{id} - (2 - Q)E_{k_s} + E_{k_s}E_{k_t}) \prod_{1 \le i < j \le n} \frac{k_j - k_i}{j - i}.$$

Notably, this generating function has already been introduced by Mills, Robbins and Rumsey in [8, Section 5]. (However, Theorem 1 provides the first formula for this generating function.)

Finally, the third generalization is a weighted enumeration of a certain type of Gelfand–Tsetlin patterns (which we denote as *weak monotone triangles* since they include monotone triangles) with prescribed bottom row, which reduces to the enumeration of monotone triangles if we take the limit $P \to 1$. (That is for $P = 1$ the weight is 1 for monotone triangles and 0 for weak monotone triangles that are not strictly increasing along all rows.) A *weak monotone*

*triangle* is a Gelfand–Tsetlin pattern $(a_{i,j})_{1 \leq i \leq j \leq n}$ with $a_{i,j-1} < a_{i-1,j}$ if $i \neq 1$ and $i < j$. The respective weight is defined as

$$\prod_{a_{i,j}:a_{i,j}<a_{i-1,j}} (P^{a_{i,j}} - [a_{i,j} = a_{i,j-1}]),$$

where [statement] $= 1$ if the statement is true and 0 otherwise.

In order to state this generating function, we need to introduce the following $P$–generalizations of the difference operator $\Delta_x := E_x - \text{id}$ and the shift operator: the $P$–difference operator is defined as ${}_P\Delta_x = P^{-x}\Delta_x$ and the $P$–shift operator is defined as ${}_PE_x = {}_P\Delta_x + \text{id}$. If we set $P = 1$ then we obtain the ordinary operators. Note that these operators commute, i.e. ${}_P\Delta_x {}_P\Delta_y = {}_P\Delta_y {}_P\Delta_x$, ${}_PE_x {}_PE_y = {}_PE_y {}_PE_x$ and ${}_PE_x {}_P\Delta_y = {}_P\Delta_y {}_PE_x$.

**Theorem 2.** *The generating function of weak monotone triangles with prescribed bottom row* $(k_1, \ldots, k_n)$, $k_1 \leq k_2 \leq \cdots \leq k_n$, *and with respect to the $P$–weight is*

$$P^{\binom{n+1}{3}} \prod_{1 \leq s < t \leq n} ({}_PE_{k_t} + {}_P\Delta_{k_s} {}_P\Delta_{k_t}) \prod_{1 \leq i < j \leq n} \frac{P^{k_j} - P^{k_i}}{P^j - P^i}.$$

The paper is organized as follows. In the following section we introduce a general recursion and a master theorem, which implies the three generalizations all at once. In Section 3 we deal with the first generalization and provide the combinatorial interpretation for (1.2) if $r(X, Y)$ is of the form given in (1.4). In Section 4 we deduce the $Q$–enumeration of monotone triangles and the $P$–enumeration of weak monotone triangles from the main theorem. Moreover, we derive some results (old and new) for the special case $Q = 2$ in this section. In Section 5 we finally prove the master theorem. In Section 6 we discuss further projects along these lines.

## 2. THE RECURSION AND THE MASTER THEOREM

We define a $PQ$–shift operator as ${}_P^Q E_x = Q\,\text{id} + {}_P\Delta_x$ and a $PQ$–identity as ${}_P^Q \text{id}_x = Q\,{}_PE_x - {}_P\Delta_x$. For $Q = 1$ we have ${}_P^Q E_x = {}_PE_x$ and ${}_P^Q \text{id}_x = \text{id}_x$ and again these operators commute with each other and also with the operators that we have introduced before the statement of Theorem 2. Moreover note that ${}_P^Q E_x p(x) = {}_P^Q \text{id}_x p(x) = Qp(x)$ if $p(x)$ is constant (with respect to $x$).

Let $S$ be a finite subset of $\mathbb{Z}^2$ and $f : S \to \mathbb{C}$ be a function. For a given $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ and a function $A(l_1, \ldots, l_{n-1})$ on $\mathbb{Z}^{n-1}$, we define the summation operator

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1})$$

associated to the pair $(S, f)$ by induction with respect to $n$. If $n = 0$ then the application of the operator gives zero, for $n = 1$ we set $\sum\limits_{(-)}^{k_1} A = A$. If $n \geq 2$ then we define

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = \left( Q^{-1}{}_P^Q E_{k_n}{}_P^Q \mathrm{id}_{k_{n-1}^*} \sum_{l_{n-1}=k_{n-1}^*}^{k_n-1} P^{l_{n-1}} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-1}) \right.$$

$$\left. + \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} \sum_{(i,j)\in S} f(i,j) \; {}_P E_{k_{n-1}}{}^i \; {}_P E_{k_{n-1}^*}{}^j A(l_1,\ldots,l_{n-3}, k_{n-1}, k_{n-1}^*)) \right|_{k_{n-1}^*=k_{n-1}} ,$$

where here and in the following $\sum\limits_{x=a}^{b} g(x) := - \sum\limits_{x=b+1}^{a-1} g(x)$ if $a > b$[1] and, for $i < 0$,

$$_P E_x^i = (\mathrm{id} + {}_P\Delta_x)^i := \left( \sum_{j=0}^{\infty} (-1)^j \; {}_P\Delta_x^j \right)^{-i}. \tag{2.1}$$

(If $i < 0$ then we will apply the operator $_P E_x^i$ only to functions $f(x)$ with $_P\Delta_x^j f(x) = 0$ for a $j \geq 0$, i.e. to functions for which the sum in (2.1) is in fact finite. Note that the operator $_P E_x^i$ specializes to $E_x^i$ for $P = 1$ also if $i < 0$.) This generalizes the summation operator from [3], where we have considered the special case $P = 1, Q = 1, S = \{(0,0)\}$ and $f(0,0) = -1$.

The $P$–binomial coefficient is defined as

$$\begin{bmatrix} x \\ m \end{bmatrix}_P = \frac{(1 - P^x)(1 - P^{x-1}) \cdots (1 - P^{x-m+1})}{(1 - P^m)(1 - P^{m-1}) \cdots (1 - P)}$$

and again we obtain the ordinary binomial coefficient $\binom{x}{m}$ if we take the limit $P \to 1$. We define $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ inductively with respect to $n$: let $\alpha_{P,Q}(1, m, S, f; k_1) = \begin{bmatrix} k_1 \\ m \end{bmatrix}_P$ and

$$\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \alpha_{P,Q}(n-1, m, S, f; l_1, \ldots, l_{n-1}).$$

In order to see that $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ is well–defined even if $S \not\subseteq \mathbb{Z}_{\geq 0}^2$ (i.e. the application of the operator $_P E_{k_j}^i$ to $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ makes sense even if $i < 0$) observe that

$$_P\Delta_x \begin{bmatrix} x \\ m \end{bmatrix}_P = \begin{bmatrix} x \\ m-1 \end{bmatrix}_P P^{-m+1} \tag{2.2}$$

and

$$\sum_{x=a}^{b} P^x \begin{bmatrix} x \\ m \end{bmatrix}_P = \sum_{x=a}^{b} P^{x+m} \; {}_P\Delta_x \begin{bmatrix} x \\ m+1 \end{bmatrix}_P = P^m \left( \begin{bmatrix} b+1 \\ m+1 \end{bmatrix}_P - \begin{bmatrix} a \\ m+1 \end{bmatrix}_P \right) \tag{2.3}$$

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[1] Note that this implies $\sum\limits_{x=a}^{a-1} g(x) = 0$.

implies by induction with respect to $m$ that $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ is a linear combination of expressions of the form $\begin{bmatrix} k_1 \\ r_1 \end{bmatrix}_P \begin{bmatrix} k_2 \\ r_2 \end{bmatrix}_P \cdots \begin{bmatrix} k_n \\ r_n \end{bmatrix}_P$ over $\mathbb{C}[P, P^{-1}, Q, Q^{-1}]$, and consequently (also by (2.2)), there is an $i \geq 0$ such that $_P\Delta^i_{k_j}\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ vanishes.

Clearly, $\alpha_{1,1}(n, 0, \{(0,0)\}, -1; k_1, \ldots, k_n)$ is equal to (1.1). Moreover, it is easy to see that $\alpha_{1,1}(n, 0, \emptyset, -; k_1, \ldots, k_n)$ is the number of Gelfand–Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$ and therefore equal to (1.3). For the general situation we will prove the following theorem.

**Theorem 3.** *Let $n$ be a positive integer, $m$ be a non–negative integer, $S \subseteq \mathbb{Z}^2$ be a finite set and $f : S \to \mathbb{C}$ be a function. Then $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$ is given by*

$$P^{\frac{1}{6}(n-1)(n^2-2n+6m)}Q^{-\binom{n}{2}} \prod_{1 \leq s < t \leq n} \left( {}^Q_P E_{k_t} \, {}^Q_P \mathrm{id}_{k_s} - {}_P\Delta_{k_s} \, {}_P\Delta_{k_t} \sum_{(i,j)\in S} f(i,j) \, {}_PE_{k_t}{}^i \, {}_PE_{k_s}{}^j \right)$$

$$\det_{1 \leq i,j \leq n} \begin{bmatrix} k_i \\ j - 1 + \delta_{j,n}m \end{bmatrix}_P.$$

*For $m = 0$ this simplifies to*

$$P^{\binom{n+1}{3}}Q^{-\binom{n}{2}} \prod_{1 \leq s < t \leq n} \left( {}^Q_P E_{k_t} \, {}^Q_P \mathrm{id}_{k_s} - {}_P\Delta_{k_s} \, {}_P\Delta_{k_t} \sum_{(i,j)\in S} f(i,j) \, {}_PE_{k_t}{}^i \, {}_PE_{k_s}{}^j \right) \prod_{1 \leq i < j \leq n} \frac{P^{k_j} - P^{k_i}}{P^j - P^i}.$$

### 3. The combinatorial interpretation of $\alpha_{1,1}(n, 0, S, f; k_1, \ldots, k_n)$

If we specialize $P = 1$, $Q = 1$ and $m = 0$ in Theorem 3 then we obtain the following formula.

$$\prod_{1 \leq s < t \leq n} \left( E_{k_t} - \Delta_{k_s}\Delta_{k_t} \sum_{(i,j)\in S} f(i,j) E_{k_t}{}^i E_{k_s}{}^j \right) \prod_{1 \leq i < j \leq n} \frac{k_j - k_i}{j - i}$$

By setting $s(X, Y) = - \sum\limits_{(i,j)\in S} f(i,j) Y^i X^j$, this is equal to (1.2) if $r(X, Y)$ is of the form given in (1.4). Thus it suffices to interpret $\alpha_{1,1}(n, 0, S, f; k_1, \ldots, k_n)$. The interpretation immediately follows from the recursion underlying this specialization, which reads as

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1}) = \sum_{l_{n-1}=k_{n-1}}^{k_n} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1, \ldots, l_{n-1})$$

$$+ \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} \sum_{(i,j)\in S} f(i,j) A(l_1, \ldots, l_{n-3}, k_{n-1} + i, k_{n-1} + j).$$

For a finite subset $S \subseteq \mathbb{Z}^2$, we define an *S–triangle* to be a triangular array $(a_{i,j})_{1 \leq i \leq j \leq n}$ (the entries are arranged in the same manner as those of monotone triangles) with the following properties: among the "inner" entries $(a_{i,j})_{1 \leq i < j < n}$ of the triangle, a certain set of non row adjacent *special entries* is identified. If $a_{i,j}$ is such a special entry then $a_{i+1,j}$ and $a_{i+1,j+1}$ are said to be the parents of this special entry and we demand that there exists an $s = (r, t) \in S$ such that $(a_{i+1,j}, a_{i+1,j+1}) = (a_{i,j}+r, a_{i,j}+t)$. We say that the special entry $a_{i,j}$ is associated to $s$.

Observe that the second summand in the recursion takes into account for these special entries. (In this case $k_{n-1}$ is the special entry.) On the other hand and regarding the first summand in the recursion, for all $a_{i,j}$ with $i \neq 1$ that are not a parent of a special entry, we demand that $a_{i-1,j-1} \leq a_{i,j} \leq a_{i-1,j}$ if $a_{i-1,j-1} \leq a_{i-1,j}$ and $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$ if $a_{i-1,j-1} > a_{i-1,j}$; in the latter case we say that the pair $(a_{i-1,j-1}, a_{i-1,j})$ is an *inversion*. We fix a function $f : S \to \mathbb{C}$ and define the weight of such an $S$–triangle as

$$(-1)^{\# \text{ of inversions}} \prod_{s \in S} f(s)^{\# \text{ of special entries associated to } s}.$$

With this definition, $\alpha_{1,1}(n, 0, S, f; k_1, \ldots, k_n)$ is the sum of the weights of all $S$–triangles with bottom row $(k_1, \ldots, k_n)$.

To give an example, observe that

$$
\begin{array}{ccccccccc}
  &   &   &   & 6  &   &   &   &   \\
  &   &   & 5 &    & 9 &   &   &   \\
  &   & 4 &   & 10 &   & 8 &   &   \\
  & 3 &   & 4 &    & 8^* &  & 9 &   \\
7 &   & 3^* & & 10 &   & 7 &   & 10
\end{array}
$$

is a $\{(0,1), (2,0)\}$–triangle, where the special entries are marked with a star. The $S$–triangle has two inversions, one in the bottom row, i.e. $(10, 7)$ and one in the middle row, i.e. $(10, 8)$. Thus, if $f(0,1) = q_1$ and $f(2,0) = q_2$ then the weight of this $S$–triangle is $q_1 q_2$.

Obviously, $\emptyset$–triangles with (weakly) increasing bottom row are simply Gelfand–Tsetlin patterns. However, the notion of $\{(0,0)\}$–triangles does not coincide with the notion of monotone triangles. Still $\alpha_{1,1}(n, 0, \{(0,0)\}, -1; k_1, \ldots, k_n)$ is the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$ if $k_1 < k_2 < \ldots < k_n$. On the one hand and as mentioned above, this easily follows from the recursion underlying $\alpha_{1,1}(n, 0, \{(0,0)\}, -1; k_1, \ldots, k_n)$. On the other hand, this can also be shown using the combinatorial interpretation that we have just introduced. For this purpose, fix $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 < k_2 < \cdots < k_n$. Under this assumption, a $\{(0,0)\}$–triangle with bottom row $(k_1, \ldots, k_n)$ has no inversion. It suffices to show that the weighted sum $\Sigma$ with respect to $f \equiv -1$ over all $\{(0,0)\}$–triangles with bottom $(k_1, \ldots, k_n)$ that have at least one special entry is the negative of the number of all Gelfand–Tsetlin patterns with bottom row $(k_1, \ldots, k_n)$ that are not monotone triangles. If we ignore the marks of the special entries of a $\{(0,0)\}$–triangle with bottom row $(k_1, \ldots, k_n)$ that appears in the sum $\Sigma$ then we clearly obtain a Gelfand–Tsetlin pattern with bottom row $(k_1, \ldots, k_n)$ that is not a monotone triangle. Thus, we have to show that for each fixed Gelfand–Tsetlin pattern that is not a monotone triangle the sum of weights of all $\{(0,0)\}$–triangles in $\Sigma$, whose unmarked version is equal to this fixed Geland–Tsetlin pattern is $-1$. Indeed, suppose that $m$ is the number of row adjacent pairs that are equal in the fixed pattern. The candidates for the special entries are those entries that are situated in a row below and in between such pairs. If we mark $k$ special entries in the Gelfand–Tsetlin pattern then the weight is $(-1)^k$ and, ignoring for a while the constraint that the special entries are not supposed to be row adjacent, there are clearly $\binom{m}{k}$ possiblities to do this. However, in this calculation the "forbidden" (i.e. row adjacent) markings cancel each other as $a_{i,j-1} = a_{i,j} = a_{i,j+1}$ implies $a_{i+1,j} = a_{i,j} = a_{i+1,j+1}$ and every forbidden marking that contains $a_{i-1,j-1}$ and $a_{i-1,j}$ but not $a_{i,j}$ cancels with the corresponding forbidden marking that includes $a_{i,j}$. Thus, as $k$ ranges between 1 and $m$, the multiplicity in

question is

$$\sum_{k=1}^{m}(-1)^k\binom{m}{k} = \sum_{k=0}^{m}(-1)^k\binom{m}{k} - 1 = (1-1)^m - 1 = -1$$

and the assertion follows. However, we want to emphasize that $\{(0,0)\}$–triangles (unlike monotone triangles) provide a combinatorial interpretation of $\alpha_{1,1}(n, 0, \{(0,0)\}, -1; k_1, \ldots, k_n)$ (and therefore of (1.1)) if $k_i \geq k_{i+1}$ for an $i \in \{1, 2, \ldots, n-1\}$.

Although we think that it is of less interest, we conclude this section by discussing the case that $Q \neq 1$, i.e. give a combinatorial interpretation of $\alpha_{1,Q}(n, 0, S, f; k_1, \ldots, k_n)$. Here, we have different requirements for the $a_{i,j}$ with $i \neq 1$ that are not a parent of a special entry. Again we demand that $a_{i-1,j-1} \leq a_{i,j} \leq a_{i-1,j}$ if $a_{i-1,j-1} \leq a_{i-1,j}$ and this entry contributes the weight $Q$ if $a_{i-1,j-1} < a_{i,j} < a_{i-1,j}$, the weight 1 if $a_{i-1,j} = a_{i,j} < a_{i-1,j}$ or $a_{i-1,j-1} < a_{i,j} = a_{i-1,j}$ and the weight $2-Q$ if $a_{i-1,j-1} = a_{i,j} = a_{i-1,j}$. If $a_{i-1,j-1} > a_{i-1,j}$ then $a_{i-1,j-1} \geq a_{i,j} \geq a_{i-1,j}$ and this entry contributes the weight $-Q$ if $a_{i-1,j-1} > a_{i,j} > a_{i-1,j}$, otherwise it contributes $1-Q$. In this case the total weight of a fixed $S$–triangle is the product of the $Q$–weights of its entries times $\prod_{s\in S} f(s)^{\#\text{ of special entries associated to } s}$.

## 4. The $Q$–enumeration of monotone triangles and the $P$–enumeration of weak monotone triangles

First we consider the $Q$–enumeration of monotone triangles. We claim that this weighted enumeration is obtained by specializing $m = 0$, $P = 1$, $S = \{(0,0)\}$ and $f \equiv -1$ in Theorem 3. Indeed, in this case the recursion simplifies to

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1, \ldots, l_{n-1})$$

$$= \left( Q^{-1}(Q\operatorname{id}+\Delta_{k_n})(QE_{k_{n-1}^*} - \Delta_{k_{n-1}^*}) \sum_{l_{n-1}=k_{n-1}^*}^{k_n-1} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1, \ldots, l_{n-1}) \right)\Bigg|_{k_{n-1}^*=k_{n-1}}$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1, \ldots, l_{n-3}, k_{n-1}, k_{n-1})$$

$$= \left( (QE_{k_{n-1}^*} + \Delta_{k_n}E_{k_{n-1}^*} - \Delta_{k_{n-1}^*}) \sum_{l_{n-1}=k_{n-1}^*}^{k_n-1} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1, \ldots, l_{n-1}) \right)\Bigg|_{k_{n-1}^*=k_{n-1}}$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1, \ldots, l_{n-3}, k_{n-1}, k_{n-1}),$$

which is furthermore equal to

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1})$$

$$= Q \sum_{l_{n-1}=k_{n-1}+1}^{k_n-1} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_n)$$

$$+ \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_{n-1}) - \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}).$$

It is straightforward to check that the specialization of the formula in Theorem 3 results in the formula of Theorem 1.

Let us report on a subtlety, which may be of importance when applying the ideas from [4] to evaluate $\alpha_{1,Q}(n,0,\{(0,0)\},-1;k_1,\ldots,k_n)$ at $(k_1,\ldots,k_n) = (1,2,\ldots,i-1,i+1,\ldots,n+1)$ in order to study a weighted refined enumeration of alternating sign matrices, see also Section 6. If we relax the condition of the strict increase of the rows of a monotone triangle to a possible weak increase in the bottom row then (1.1) is the number of monotone triangles with bottom row $(k_1,\ldots,k_n)$ if $k_1 \le k_2 \le \ldots \le k_n$. However, this does not generalize to the $Q$–enumeration of monotone triangle as the recursion simplifies to

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = (2-Q) \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_{n-1})$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1})$$

if $k_{n-1} = k_n$ and not to

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_{n-1})$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1}).$$

Before we turn to the $P$–enumeration of weak monotone triangles, we will demonstrate how several 2–enumeration of alternating–sign–matrix–structures follow from the $Q$–generating function. We start with the 2–enumeration of ordinary $n \times n$ alternating sign matrices (with respect to the number of $-1$s in the alternating sign matrix), which was first established by Mills, Robbins and Rumsey in [8] as a corollary of their Theorem 2 (they have shown that it is given by $2^{\binom{n}{2}}$) long before the ordinary enumeration was finally settled. Also our derivation shows that the 2–enumeration is of a much simpler nature than the ordinary enumeration.

Indeed, the generating function from Theorem 1 simplifies to

$$\prod_{1\leq s<t\leq n}(\mathrm{id}+E_{k_s}E_{k_t})\prod_{1\leq i<j\leq n}\frac{k_j-k_i}{j-i}$$

in this case. Now, the crucial fact is that the operator $\prod_{1\leq s<t\leq n}(\mathrm{id}+E_{k_s}E_{k_t})$ is symmetric in $k_1,k_2,\ldots,k_n$ and, consequently, Lemma 1 from [4] implies that the generation function is equal to $P(1,1,\ldots,1)\prod_{1\leq i<j\leq n}\frac{k_j-k_i}{j-i}$, where $P(X_1,X_2,\ldots,X_n)=\prod_{1\leq s<t\leq n}(\mathrm{id}+X_sX_t)$ and, therefore, equal to

$$2^{\binom{n}{2}}\prod_{1\leq i<j\leq n}\frac{k_j-k_i}{j-i}.\tag{4.1}$$

(This result has already appeared implicitly in [9].) The 2–enumeration is obtained by setting $k_i=i$ in this formula. If we specialize $(k_1,\ldots,k_{n-1})=(1,2,\ldots,l-1,l+1,\ldots,n)$ in the formula for the 2–enumeration of monotone triangles with bottom row $(k_1,k_2,\ldots,k_{n-1})$ then we obtain the 2–enumeration of $n\times n$ alternating sign matrices where the unique 1 in the first row is in the $l$–th column. That is

$$2^{\binom{n-1}{2}}\prod_{\substack{1\leq i<j\leq n\\i,j\neq l}}(j-i)\prod_{1\leq i<j\leq n-1}\frac{1}{j-i}=2^{\binom{n-1}{2}}\prod_{1\leq i<j\leq n}(j-i)\prod_{j=l+1}^{n}\frac{1}{j-l}\prod_{i=1}^{l-1}\frac{1}{l-i}\prod_{1\leq i<j\leq n-1}\frac{1}{j-i}$$

$$=2^{\binom{n-1}{2}}\prod_{i=1}^{n-1}(n-i)\frac{1}{(n-l)!(l-1)!}=2^{\binom{n-1}{2}}\binom{n-1}{l-1}.$$

Similarly, one can reprove the 2–enumeration of $(2n-1)\times(2n-1)$ vertically symmetric alternating sign matrices by specializing $(k_1,k_2,\ldots,k_n)=(1,3,\ldots,2n-1)$, which results in $2^{(n-1)(n-2))}$, see [6].

In fact, it is possible to generalize the 2–enumeration of $n\times n$ alternating sign matrices to certain partial alternating sign matrices. Let a partial $m\times n$ alternating sign matrix be an $m\times n$ matrix with entries $0,1,-1$, where the entries $1$ and $-1$ alternate in each row and column, each row sum is 1 and the first non–zero entry of each column is 1 if there is any. (Such objects only exist if $m\leq n$.) The well–known bijection between alternating sign matrices and monotone triangles shows that partial $m\times n$ alternating sign matrices are in bijection with monotone triangles with $m$ rows such that $1\leq k_1<k_2<\ldots<k_m\leq n$. Thus and by (4.1), the 2–enumeration of these objects is given by

$$2^{\binom{m}{2}}\sum_{1\leq k_1<k_2<\ldots<k_m\leq n}\prod_{1\leq i<j\leq m}\frac{k_j-k_i}{j-i}=2^{\binom{m}{2}}\sum_{0\leq x_1\leq x_2<\ldots\leq x_m\leq n-m}\prod_{1\leq i<j\leq m}\frac{x_j-x_i+j-i}{j-i}.\tag{4.2}$$

As $\prod_{1\leq i<j\leq m}\frac{x_j-x_i+j-i}{j-i}$ is the number of semistandard tableaux of shape $(x_m,x_{m-1},\ldots,x_1)$, the sum on the right hand side of (4.2) is the number of columnstrict plane partitions with at most $n-m$ columns and parts in $\{1,2,\ldots,m\}$. By the Bender–Knuth (ex–)Conjecture, see for instance [2], this number is equal to $\prod_{i=1}^{m}\frac{(n-m+i)_i}{(i)_i}$ where $(a)_n=a(a+1)\cdots(a+n-1)$, and

thus the 2–enumeration of partial $m \times n$ alternating sign matrices is given by

$$2^{\binom{m}{2}} \prod_{i=1}^{m} \frac{(n-m+i)_i}{(i)_i}.$$

For the $P$–enumeration of weak monotone triangles, the weighted enumeration is obtained by specializing $m = 0$, $Q = 1$, $S = \{(0,0)\}$ and $f \equiv -1$ in Theorem 3, as the recursion simplifies to

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1,\ldots,l_{n-1}) = {}_PE_{k_n} \sum_{l_{n-1}=k_{n-1}}^{k_n-1} P^{l_{n-1}} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-1})$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1})$$

$$= \sum_{l_{n-1}=k_{n-1}}^{k_n-1} P^{l_{n-1}} \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-1}) + \sum_{(l_1,\ldots,l_{n-2})}^{(k_1,\ldots,k_{n-1})} A(l_1,\ldots,l_{n-2},k_n)$$

$$- \sum_{(l_1,\ldots,l_{n-3})}^{(k_1,\ldots,k_{n-2})} A(l_1,\ldots,l_{n-3},k_{n-1},k_{n-1})$$

in this case. This easily implies that $\alpha_{P,1}(n,0,\{(0,0)\},-1;k_1,\ldots,k_n)$ is the weighted enumeration of weak monotone triangles with bottom row $(k_1,\ldots,k_n)$ and with respect to the $P$–weight, which was defined in the introduction. Clearly, the formula in Theorem 3 simplifies to the formula in Theorem 2 if we set $m = 0$, $Q = 1$, $S = \{(0,0)\}$ and $f \equiv -1$.

## 5. Proof of Theorem 3

For fixed $S \subseteq \mathbb{Z}^2$ and $f : S \to \mathbb{C}$, we define the operator

$$V_{x,y} = {}_P^Q E_x \, {}_P^Q \mathrm{id}_y - {}_P\Delta_x \, {}_P\Delta_y \sum_{(i,j)\in S} f(i,j) \, {}_PE_x{}^i \, {}_PE_y{}^j.$$

The following lemma relates this operator to the summation operator defined in Section 2.

**Lemma 1.** *Suppose $B(l_1,\ldots,l_{n-1})$ is a function in $l_1,\ldots,l_{n-1}$ such that for all $i \in \{1,2,\ldots,n-2\}$ we have $V_{l_i,l_{i+1}}B(l_1,\ldots,l_{n-1}) = 0$ if $l_i = l_{i+1}$. Then*

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} {}_P\Delta_{l_1} \cdots {}_P\Delta_{l_{n-1}} B(l_1,\ldots,l_{n-1})$$

$$= \sum_{r=1}^{n}(-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q\mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}_P^Q E_{k_t} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_n).$$

*Proof.* We use induction with respect to $n$. For $n = 0$ and $n = 1$ there is nothing to prove. Suppose that $n \geq 2$. Then, by definition and by the induction hypothesis, we have

$$
\sum_{\substack{(l_1,\ldots,l_{n-1})}}^{(k_1,\ldots,k_n)} {}_P\Delta_{l_1} \cdots {}_P\Delta_{l_{n-1}} B(l_1,\ldots,l_{n-1}) = \left( Q^{-1} {}_P^Q E_{k_n} {}_P^Q \mathrm{id}_{k_{n-1}^*} \sum_{l_{n-1}=k_{n-1}^*}^{k_n-1} P^{l_{n-1}} {}_P\Delta_{l_{n-1}} \right.
$$

$$
\sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}_P^Q E_{k_t} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-1},l_{n-1})
$$

$$
+ \sum_{r=1}^{n-2} (-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}_P^Q E_{k_t} \sum_{(i,j)\in S} f(i,j) \, {}_P E_{k_{n-1}}{}^i \, {}_P E_{k_{n-1}^*}{}^j
$$

$$
\left. {}_P\Delta_{k_{n-1}} \, {}_P\Delta_{k_{n-1}^*} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-2},k_{n-1},k_{n-1}^*) \right)\Big|_{k_{n-1}^*=k_{n-1}} .
$$

This is obviously equal to

$$
\sum_{r=1}^{n-1} (-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}_P^Q E_{k_t} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-1},k_n)
$$

$$
+ \sum_{r=1}^{n-1} (-1)^r \left( \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n-1} {}_P^Q E_{k_t} {}_P^Q \mathrm{id}_{k_{n-1}^*} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-1},k_{n-1}^*) \right)\Big|_{k_{n-1}^*=k_{n-1}}
$$

$$
+ \sum_{r=1}^{n-2} (-1)^r \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}_P^Q E_{k_t} \left( (V_{k_{n-1},k_{n-1}^*} - {}_P^Q E_{k_{n-1}} {}_P^Q \mathrm{id}_{k_{n-1}^*}) \right.
$$

$$
\left. B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-2},k_{n-1},k_{n-1}^*) \right)\Big|_{k_{n-1}^*=k_{n-1}} .
$$

If we move the $(n-1)$–st summand of the second sum to the first sum and combine the third sum with the remainder of the second sum then we see that this is furthermore equal to

$$
\sum_{r=1}^{n} (-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}_P^Q E_{k_t} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-1},k_n)
$$

$$
+ \sum_{r=1}^{n-2} (-1)^r \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n-2} {}_P^Q E_{k_t} (V_{k_{n-1},k_{n-1}^*} B(k_1,\ldots,k_{r-1},k_{r+1},\ldots,k_{n-2},k_{n-1},k_{n-1}^*)) \Big|_{k_{n-1}^*=k_{n-1}}
$$

and the assertion follows. $\qquad\square$

In the following lemma we use the previous lemma to apply the summation operator to a special $B(l_1,\ldots,l_{n-1})$.

**Lemma 2.** *Let $m_2, m_3, \ldots, m_n \geq 0$ be integers and set $m_1 = -1$. Then*

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) \det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1} \end{bmatrix}_P$$

$$= Q^{-n+1} \prod_{j=2}^{n} P^{m_j} \left( \prod_{1 \leq s < t \leq n} V_{k_t,k_s} \right) \det_{1 \leq i,j \leq n} \begin{bmatrix} k_i \\ m_j + 1 \end{bmatrix}_P.$$

*Proof.* Let

$$B(l_1, \ldots, l_{n-1}) = \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) \det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1} + 1 \end{bmatrix}_P.$$

First we show that $B(l_1, \ldots, l_{n-1})$ has the property, which is presumed in Lemma 1. For that purpose it suffices to show that

$$(\mathrm{id} + S_{l_i,l_{i+1}}) V_{l_i,l_{i+1}} B(l_1, \ldots, l_{n-1}) = 0$$

for all $i \in \{1, 2, \ldots, n-2\}$, where $S_{x,y}$ is the *swapping operator*, defined as $S_{x,y} f(x, y) = f(y, x)$. This assertion follows, since, on the one hand,

$$V_{l_i,l_{i+1}} V_{l_{i+1},l_i} \left( \prod_{\substack{1 \leq s < t \leq n-1 \\ (s,t) \neq (i,i+1)}} V_{l_t,l_s} \right)$$

is symmetric in $l_i$ and $l_{i+1}$ and therefore commutes with $S_{l_i,l_{i+1}}$ and, on the other hand, $\det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1}+1 \end{bmatrix}_P$ is antisymmetric in $l_i$ and $l_{i+1}$. Furthermore, by (2.2), we conclude that

$$_P\Delta_{l_1} \cdots {}_P\Delta_{l_{n-1}} B(l_1, \ldots, l_{n-1}) = {}_P\Delta_{l_1} \cdots {}_P\Delta_{l_{n-1}} \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) \det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1} + 1 \end{bmatrix}_P$$

$$= \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) {}_P\Delta_{l_1} \cdots {}_P\Delta_{l_{n-1}} \det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1} + 1 \end{bmatrix}_P$$

$$= \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) \det_{1 \leq i,j \leq n-1} {}_P\Delta_{l_i} \begin{bmatrix} l_i \\ m_{j+1} + 1 \end{bmatrix}_P = \prod_{j=2}^{n} P^{-m_j} \left( \prod_{1 \leq s < t \leq n-1} V_{l_t,l_s} \right) \det_{1 \leq i,j \leq n-1} \begin{bmatrix} l_i \\ m_{j+1} \end{bmatrix}_P.$$

Therefore, by Lemma 1, the left–hand side of the identity stated in the lemma is equal to

$$\prod_{j=2}^{n} P^{m_j} \sum_{r=1}^{n} (-1)^{r-1} \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}_P^Q E_{k_t} B(k_1, \ldots, k_{r-1}, k_{r+1}, \ldots, k_n).$$

By the definition of $B(l_1, \ldots, l_{n-1})$, this is equal to

$$\prod_{j=2}^{n} P^{m_j} \sum_{r=1}^{n} (-1)^{r-1} \left( \prod_{\substack{1 \le s < t \le n \\ s,t \ne r}} V_{k_t,k_s} \right) \prod_{s=1}^{r-1} {}_P^Q \mathrm{id}_{k_s} \prod_{t=r+1}^{n} {}_P^Q E_{k_t}$$

$$\det_{1 \le i,j \le n-1} \left[ \begin{matrix} l_i \\ m_{j+1}+1 \end{matrix} \right]_P \bigg|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,\widehat{k_r},\ldots,k_n)} .$$

Since

$$_P\Delta_{k_r} \det_{1 \le i,j \le n-1} \left[ \begin{matrix} l_i \\ m_{j+1}+1 \end{matrix} \right]_P \bigg|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,\widehat{k_r},\ldots,k_n)} = 0,$$

this is furthermore equal to

$$Q^{-n+1} \prod_{j=2}^{n} P^{m_j} \sum_{r=1}^{n} (-1)^{r-1} \left( \prod_{\substack{1 \le s < t \le n \\ s,t \ne r}} V_{k_t,k_s} \right) \prod_{s=1}^{r-1} V_{k_r,k_s} \prod_{t=r+1}^{n} V_{k_t,k_r}$$

$$\det_{1 \le i,j \le n-1} \left[ \begin{matrix} l_i \\ m_{j+1}+1 \end{matrix} \right]_P \bigg|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,\widehat{k_r},\ldots,k_n)}$$

$$= Q^{-n+1} \prod_{j=2}^{n} P^{m_j} \sum_{r=1}^{n} (-1)^{r-1} \left( \prod_{1 \le s < t \le n} V_{k_t,k_s} \right) \det_{1 \le i,j \le n-1} \left[ \begin{matrix} l_i \\ m_{j+1}+1 \end{matrix} \right]_P \bigg|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,\widehat{k_r},\ldots,k_n)}$$

$$= Q^{-n+1} \prod_{j=2}^{n} P^{m_j} \left( \prod_{1 \le s < t \le n} V_{k_t,k_s} \right) \sum_{r=1}^{n} (-1)^{r-1} \det_{1 \le i,j \le n-1} \left[ \begin{matrix} l_i \\ m_{j+1}+1 \end{matrix} \right]_P \bigg|_{(l_1,\ldots,l_{n-1})=(k_1,\ldots,\widehat{k_r},\ldots,k_n)}$$

This is now the right–hand side of the identity in the statement of the lemma, which is evident when expanding the determinant in the statement of the lemma with respect to the first column. $\square$

We define a quantity that is even more general than $\alpha_{P,Q}(n, m, S, f; k_1, \ldots, k_n)$. For $r \ge 1$ and $(m_1, \ldots, m_r) \in \mathbb{Z}_{\ge 0}^r$, let

$$\alpha_{P,Q}(1, (m_1, \ldots, m_r), S, f; k_1, \ldots, k_r) = \left( \prod_{1 \le s < t \le r} V_{k_t,k_s} \right) \det_{1 \le i,j \le r} \left[ \begin{matrix} k_i \\ m_j \end{matrix} \right]_P$$

and, for $n > 1$, let

$$\alpha_{P,Q}(n, (m_1, \ldots, m_r), S, f; k_1, \ldots, k_{n+r-1})$$

$$= \sum_{(l_1,\ldots,l_{n+r-2})}^{(k_1,\ldots,k_{n+r-1})} \alpha_{P,Q}(n-1, (m_1, \ldots, m_r), S, f; l_1, \ldots, l_{n+r-2}).$$

By induction with respect to $n$, Lemma 2 shows that $\alpha_{P,Q}(n, (m_1, \ldots, m_r), S, f; k_1, \ldots, k_{n+r-1})$ is equal to

$$P^{\frac{1}{6}(n+3r-3)(n-1)(n-2)+(m_1+m_2+\ldots+m_r)(n-1)} Q^{-\binom{n}{2}} \left( \prod_{1 \leq s < t \leq n+r-1} V_{k_t, k_s} \right)$$

$$\det_{1 \leq i,j \leq n+r-1} \left[ \begin{array}{c} k_i \\ [j < n](j-1) + [j \geq n](m_{j-n+1} + n - 1) \end{array} \right]_P .$$

The first statement in Theorem 3 is the special case $r = 1$ and $m_1 = m$. The second statement follows as

$$\det_{1 \leq i,j \leq n} \left[ \begin{array}{c} k_i \\ j-1 \end{array} \right]_P = P^{\binom{n}{2}} \prod_{1 \leq i < j \leq n} \frac{P^{k_j} - P^{k_i}}{P^j - P^i}$$

by the $q$–Vandermonde determinant evaluation.

## 6. REMARKS AND FURTHER PROJECTS

The starting point for this paper was [3], where we have studied the recursion underlying the counting function for monotone triangles with prescribed bottom row. In the present paper, we have considered a generalized recursion, which we have obtained by carefully introducing a number of new parameter, namely $m, P, Q$, a finite subset $S \subseteq \mathbb{Z}^2$ and a function $f : S \to \mathbb{C}$, in the original recursion. The analysis of this generalized recursion was possible as we have finally noticed that the analysis of the original recursion can be simplified significantly.

With the exception of $m$, all new parameters have been used to either obtain weighted enumerations of monotone triangles, respectively weak monotone triangles or a combinatorial interpretation of a generalization of (1.1). However, the parameter $m$ is also of special interest since it offers the possibility to "control" top and bottom row of a monotone triangle – so far we were only able to "control" either row. Indeed, for fixed $n \geq 1$ and $1 \leq i \leq n$, let $(c_{p,q})_{p,q\geq0}$ be complex coefficients, where almost all coefficients are zero, such that the polynomial $\sum_{p,q\geq0} c_{p,q} \binom{k-p}{q}$ (in $k$) vanishes for all $k \in \{1, 2, \ldots, n\} \setminus \{i\}$ and is equal to 1 for $k = i$. Then, it is not hard to see that

$$\sum_{p,q\geq0} c_{p,q}\, \alpha_{1,1}(n, q, \{(0,0)\}, -1; k_1 - p, \ldots, k_n - p)$$

is the number of monotone triangles with bottom row $(k_1, \ldots, k_n)$ and top row $i$. (Here, we need the fact that

$$\sum_{(l_1,\ldots,l_{n-1})}^{(k_1,\ldots,k_n)} A(l_1 - p, l_2 - p, \ldots, l_{n-1} - p) = \sum_{(l_1,\ldots,l_{n-1})}^{(k_1-p,\ldots,k_n-p)} A(l_1, l_2, \ldots, l_{n-1}).)$$

In a forthcoming paper, we want to use the methods from [4] to attack a certain doubly refined enumeration of alternating sign matrices, namely study the number of $n \times n$ alternating sign matrices where the unique 1 in the top row is in column $i$ and the unique 1 in the bottom row is in column $j$. (Cleary, this number is equal to

$$\sum_{p,q\geq0} c_{p,q}\, \alpha_{1,1}(n-1, q, \{(0,0)\}, -1; 1-p, 2-p, \ldots, j-1-p, j+1-p, j+2-p, \ldots, n-p).)$$

Note that this doubly refined enumeration of alternating sign matrices has already been considered in [10]. Promising computerexperiments show that $\alpha_{1,1}(n-1, q, \{(0,0)\}, -1; 1-p, \ldots, j-1-p, j+1-p, \ldots, n-p)$ is in fact "round" (i.e. has only relatively small prime factors) for certain choices of $p$ and $q$. For instance, we have worked out the following conjecture for $q = 1$,

$$\alpha_{1,1}(n-1, 1, \{(0,0)\}, -1; 1-p, 2-p, \ldots, j-1-p, j+1-p, j+2-p, \ldots, n-p)$$
$$= (j-p)A_{n-1,j} + A_n \frac{(n-j+1)_{2j-3} n(n-2j+1)(n+j-1)}{(2n-j-1)(2n-j+1)_{j-1}(j-1)!},$$

where $A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ is the number of $n \times n$ alternating sign matrices and

$$A_{n,i} = \binom{n+i-2}{i-1} \frac{(2n-i-1)!}{(n-i)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

is the number of $n \times n$ alternating sign matrices that have a 1 which is situated in the first row and $i$–th column.

Of course, it is also of interest to apply the ideas from [4] to obtain informations on the special evaluations at $(k_1, \ldots, k_n) = (1, 2, \ldots, i-1, i+1, \ldots, n+1)$ of the other generalizations of (1.1). For instance, the evaluation of the $Q$–enumeration in Theorem 1 at $(k_1, \ldots, k_n) = (1, 2, \ldots, n)$ is the weighted enumeration of $n \times n$ alternating sign matrices with respect to the number of $-1$s in the alternating sign matrix and, similary, the evaluation at $(k_1, \ldots, k_n) = (1, 3, \ldots, 2n-1)$ is the weighted enumeration of $(2n-1) \times (2n-1)$ vertically symmetric alternating sign matrices. So far, there do not exist formulas for these generating functions (it is likely that it is rather difficult to come up with a simple formula as the generating functions do not seem to be "round"), however Kuperberg [6, Theorem 4] proved a number of factorizations regarding generating functions of this type. Is it possible to use the methods from [4] to deduce refined versions of these relations?

Another natural question is whether the new insight in the recursion underlying (1.1) leads to further generalizations of the formula. A project along these lines will be the following: for $1 \le i \le n$, let a monotone $(i, n)$–trapezoid be a monotone triangle with the first $i-1$ rows removed. We want to study the number of monotone $(i, n)$–trapezoids with prescribed bottom row $(k_1, \ldots, k_n)$. Obviously, the underlying recursion is the same as those for monotone triangles with prescribed bottom row. The difference (and the difficulty) lies in the inital condition. Secondly, we want to remark that our $P$–enumeration of weak monotone triangles with prescribed bottom row is a result of our efforts to obtain a weighted enumeration of monotone triangles, where the weight of a given monotone triangle is equal (or at least related) to $P$ raised to the power of the sum of entries. Is it possible to write down a closed (operator) formula for a generating function of this type, or is the $P$–enumeration of weak monotone triangles already the best we can achieve in this respect? Thirdly, it should be mentionend that the result regarding the combinatorial interpretation of (1.2) emerged during our efforts to find other objects whose counting functions can be expressed by an operator formula. The (somehow inverse) strategy, which finally led to the result, was to start with an operator formula and to search for objects whose weighted enumeration is given by the formula. The solution given in Section 3 is in the sense not satisfactory as it does not lead to a "plain"

enumeration when specializing the weights and, moreover, the definition of $S$–triangles is a bit involved. Clearly, it would be of interest to search for other operators formulas and (simpler) combinatorial objects that are enumerated by these formulas by varying the operators and/or the (factorizing) polynomial to which the operators are applied.

## References

[1] D. M. Bressoud, *Proofs and Confirmations, The Story of the Alternating Sign Matrix Conjecture*, Cambridge University Press, Cambridge, 1999.

[2] I. Fischer, A method for proving polynomial enumeration formulas, *J. Combin. Theory Ser. A* **111** (2005), 37 – 58.

[3] I. Fischer, The number of monotone triangles with prescribed bottom row, *Adv. in Appl. Math.* **37** (2006), no. 2, 249 – 267.

[4] I. Fischer, A new proof of the refined alternating sign matrix theorem, *J. Combin. Theory Ser. A* **114** (2007), no. 2, 253 – 264.

[5] I. M. Gelfand and M. L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices (in Russian), *Doklady Akad. Nauk. SSSR (N. S.)* **71** (1950), 825 – 828.

[6] G. Kuperberg, Symmetry classes of alternating-sign matrices under one roof, *Ann. of Math. (2)* **156** (2002), no. 3, 835-866.

[7] W. H. Mills, D. P. Robbins and H. Rumsey, Proof of the Macdonald conjecture, *Invent. Math.* **66** (1982), 73 – 87.

[8] W. H. Mills, D. P. Robbins and H. Rumsey, Alternating sign matrices and descending plane partitions, *J. Combin. Theory Ser. A* **34** (1983), 340 – 359.

[9] N. Elkies, G. Kuperberg, M. Larsen and J. Propp, Alternating sign matrices and domino tilings, *J. Algebraic Combin.* **1** (1992), no. 2, 111–132; no. 3, 219-234.

[10] Yu. G. Stroganov, A new way to deal with Izergin-Korepin determinant at root of unity, `arXiv:math-ph/0204042`.

[11] R. P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge University Press, Cambridge 1999.

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